

Asymptotic properties of gravitational and electromagnetic fields in higher dimensions

Marcello Ortaggio, Alena Pravdová

Institute of Mathematics, Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Prague 1, Czech Republic

E-mail: ortaggio@math.cas.cz, pravdova@math.cas.cz

Abstract.

We summarize the fall-off of electromagnetic and gravitational fields in $n > 5$ dimensional Ricci-flat spacetimes along an asymptotically expanding non-singular geodesic null congruence.

1. Introduction

Under suitable assumptions, the well-known peeling-off property characterizes the behavior of the gravitational and electromagnetic fields at null infinity (see, e.g., [1, 2] and references therein). It has been observed [3] that the Weyl tensor peels off differently in $n > 4$ dimensions. Here, we summarize our recent results [4, 5] on the leading-order behavior of gravitational and electromagnetic fields in higher dimensions. Ref. [4] partly recovers the results of [3] but uses a different method and different assumptions. We restrict to Ricci-flat spacetimes with suitable properties at null infinity (a cosmological constant can be included [4, 5]), formulated in terms of a geodesic null vector field $\ell = \partial_r$ (r is an affine parameter) and of the Weyl tensor, using a “null” frame [6] based on two null vectors $\mathbf{m}_{(0)} = \ell$, $\mathbf{m}_{(1)} = \mathbf{n}$ and $n - 2$ orthonormal spacelike vectors $\mathbf{m}_{(i)}$ ($i, j, \dots = 2, \dots, n - 1$). First, we assume that the optical matrix $\rho_{ij} = \ell_{a;b} m_{(i)}^a m_{(j)}^b$ is *asymptotically non-singular and expanding* [4, 5] (this includes asymptotically flat spacetimes [3] but also holds more generally – see [7] in four dimensions). Furthermore, we assume that the boost-weight (b.w.) +2 Weyl components $\Omega_{ij} \equiv C_{0i0j} = C_{abcd} \ell^a m_{(i)}^b \ell^c m_{(j)}^d$ fall off as

$$\Omega_{ij} = O(r^{-\nu}) \quad (\nu > 2). \quad (1)$$

Again, this is satisfied in asymptotically flat spacetimes [3] (e.g., $\Omega_{ij} = O(r^{-5})$ in the 4D spacetimes of [7]). Under the above conditions, one is able to determine how the Maxwell and Weyl tensors fall off as $r \rightarrow \infty$, as we summarize in sections 2 and 3. However, as an intermediate step, one also needs the r -dependence of the Ricci rotation coefficients and of the derivative operators [6], which is given in [4] (it follows from the Ricci identities [8], also using the commutators [9] and the Bianchi identities [10]). For example, $\rho_{ij} = \frac{\delta_{ij}}{r} + \dots$. For brevity, in this paper, we discuss only results in $n > 5$ dimensions – the case $n \geq 5$ is studied in [4, 5].

2. Electromagnetic field

We start from the simpler case of *test* Maxwell fields in the background of an n -dimensional Ricci-flat spacetime satisfying the assumptions of section 1 [5]. The gravitational field (Weyl tensor)

can be treated similarly, however, resulting in a larger number of possible cases (section 3).

In the frame of section 1, we assume that for $r \rightarrow \infty$ the Maxwell components have a *power-like* behavior described by

$$F_{0i} = O(r^\alpha), \quad F_{01} = O(r^\beta), \quad F_{ij} = O(r^\gamma), \quad F_{1i} = O(r^\delta). \quad (2)$$

The empty-space Maxwell equations $F^a_{b;a} = 0 = F_{[ab;c]}$ (see [5, 11] for their GHP and NP form) determine the possible values of α , β , γ and δ . We assume that if a generic component f behaves as $f = O(r^{-\zeta})$ then $\partial_r f = O(r^{-\zeta-1})$ and $\partial_A f = O(r^{-\zeta})$. As it turns out, α can be chosen arbitrarily, giving rise to two main cases, $\alpha \geq -2$ or $\alpha < -2$. In the latter, one needs to choose whether $\gamma \geq -2$ or $\gamma < -2$, and then specify more precisely the value of α , as we detail.

2.1. Case $\alpha \geq -2$.

In this case, all components fall off at the same speed, i.e.,

$$F_{0i} = O(r^\alpha), \quad F_{01} = O(r^\alpha), \quad F_{ij} = O(r^\alpha), \quad F_{1i} = O(r^\alpha). \quad (3)$$

The electromagnetic field does not peel. This describes, e.g., a uniform magnetic field permeating asymptotically flat black holes [12] (or black rings [13] if $n = 5$ is included, cf. [5]).

2.2. Case $\alpha < -2$.

Generically, we have

$$F_{0i} = O(r^\alpha), \quad (4)$$

$$F_{01} = o(r^{-2}), \quad F_{ij} = O(r^{-2}), \quad (5)$$

$$F_{1i} = O(r^{-2}). \quad (6)$$

The above behavior includes the special case when ℓ is an aligned null direction of the Maxwell field, i.e., $F_{0i} = 0$ (in the formal limit $\alpha \rightarrow -\infty$). The leading term is of type II. Examples can be obtained as a “linearized” Maxwell field limit of certain full Einstein-Maxwell solutions given in [14] for even n . Several subcases are possible when $\gamma < -2$.

2.2.1. Subcase (a): $\gamma < -2$ with $1 - \frac{n}{2} \leq \alpha < -2$. In this case, one has the same results as in section 2.1 above. This subcase does not exist for $n = 6$.

2.2.2. Subcase (b): $\gamma < -2$ with $-\frac{n}{2} \leq \alpha < 1 - \frac{n}{2}$. Here, we have

$$F_{0i} = O(r^\alpha), \quad (7)$$

$$F_{01} = O(r^\alpha), \quad F_{ij} = O(r^\alpha), \quad (8)$$

$$F_{1i} = O(r^{1-n/2}). \quad (9)$$

The leading term falls off as $1/r^{\frac{n}{2}-1}$ and is of type N. This is characteristic of radiative fields (note that $T_{11} \propto F_{1i}F_{1i} \sim 1/r^{n-2}$ and the energy flux along ℓ can be directly related to the energy loss, at least in the case of asymptotically flat spacetimes – cf. [15–17] for $n = 4$). As opposed to the well-known four-dimensional case, here, ℓ cannot be aligned with F_{ab} if radiation is present (since $\alpha \geq -\frac{n}{2}$). In the case $\alpha = -\frac{n}{2}$, if one assumes that F_{1i} has a power-like behavior also at the subleading order, from the Maxwell equations, one finds $F_{1i} = F_{1i}^{(0)} r^{1-\frac{n}{2}} + O(r^{-n/2})$, which gives the peeling-off behavior

$$F_{ab} = \frac{N_{ab}}{r^{\frac{n}{2}-1}} + \frac{G_{ab}}{r^{\frac{n}{2}}} + \dots \quad \left(\alpha = -\frac{n}{2} \right). \quad (10)$$

The subleading term is algebraically general, which is qualitatively different from the 4D case [1, 2, 16, 17]. This resembles the behavior of the Weyl tensor of higher dimensional asymptotically flat spacetimes [3]. See [5] for a possible different peeling-off in five dimensions.

2.2.3. *Subcase (c): $\gamma < -2$ with $2 - n \leq \alpha < -\frac{n}{2}$.* The same results as in section 2.1 apply.

2.2.4. *Subcase (d): $\gamma < -2$ with $\alpha < 2 - n$.* We have

$$F_{0i} = O(r^\alpha), \quad (11)$$

$$F_{01} = O(r^{2-n}), \quad F_{ij} = o(r^{2-n}), \quad (12)$$

$$F_{1i} = O(r^{2-n}). \quad (13)$$

The leading term is of type II and falls off as $1/r^{n-2}$ (it is purely electric in the subcase $F_{1i} = o(r^{2-n})$). This behavior includes the Coulomb field of a weakly charged asymptotically flat black hole [12, 18] (or black ring [13] if $n = 5$ is included [5]). In the special subcase $F_{01} = o(r^{2-n})$, the same results as in section 2.1 again apply (for example, for $n = 5$ and $\alpha = -4$, this is the case of the weak-field limit of the 5D dipole black rings of [19]).

Let us observe that in all cases, type N fields for which ℓ is aligned are not permitted [11, 20].

2.3. The case of p -forms

The above results for a 2-form F_{ab} can be extended easily [5] to p -form fields satisfying the generalized Maxwell equations (given in [11] in the GHP notation). In *even* dimensions, the special case $p = n/2$ (including $n = 4$, $p = 2$) has unique properties. It peels off as

$$F_{a_1 \dots a_p} = \frac{N_{a_1 \dots a_p}}{r^{\frac{n}{2}-1}} + \frac{II_{a_1 \dots a_p}}{r^{\frac{n}{2}}} + \dots \quad \left(p = \frac{n}{2}\right). \quad (14)$$

The (radiative) leading term is of type N and falls off as $1/r^{\frac{n}{2}-1}$. In contrast to the case $p = 2$ discussed above (or, in fact, any other $p \neq n/2$), Maxwell fields of type N aligned with ℓ are now permitted [5] and the peeling (14) applies also in the presence of a cosmological constant [5]. Corresponding solutions of the *full Einstein-Maxwell equations* have recently been obtained [21].

3. Gravitational field

The method to be used for the Weyl tensor [4] is essentially similar, now $-\nu$ playing the role that α played above. Instead of the Maxwell equations, one has to integrate the system “Bianchi-Ricci-commutators”. However, there is now extra freedom in the choice of possible boundary conditions. In particular, three possible choices for the behavior of b.w. $+1$ components Ψ_{ijk} are possible (cases (i), (ii) and (iii) below). Once the fall-off of Ω_{ij} and Ψ_{ijk} has been specified, the next step is to determine the fall-off of the b.w. 0 components Φ_{ijkl}

$$\Phi_{ijkl} = O(r^{\beta_c}). \quad (15)$$

The parameter β_c can then be used to label various possible subcases, which we now present.

3.1. Case (i): $\Omega_{ij} = O(r^{-\nu})$, $\Psi_{ijk} = O(r^{-\nu})$

In all cases given here, we have (this will not be repeated every time below)

$$\Omega_{ij} = O(r^{-\nu}) \quad (\nu > 2), \quad \Psi_{ijk} = O(r^{-\nu}). \quad (16)$$

3.1.1. *Subcase (A): $\beta_c = -2$.* In this case, necessarily $\beta_c > -\nu$ and we have the following possible behaviors, depending on how ν is chosen (cf. [4] for a few further special subcases):

A1:

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= o(r^{-2}), & \Phi_{ij}^A &= o(r^{-2}) & (2 < \nu \leq 3), \\ \Psi'_{ijk} &= O(r^{-2}), \\ \Omega'_{ij} &= O(r^\sigma) & (-2 \leq \sigma < -1); \end{aligned} \quad (17)$$

A2:

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{-3}), & \Phi &= O(r^{-\nu}), & \Phi_{ij}^A &= O(r^{-3}) & (3 < \nu < 4), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{-3}), \\ \Omega'_{ij} &= O(r^{-2}); \end{aligned} \quad (18)$$

A3:

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{-3}), & \Phi &= O(r^{-4}), & \Phi_{ij}^A &= O(r^{-3}) & (\nu \geq 4), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{-3}), \\ \Omega'_{ij} &= O(r^{-2}), \end{aligned} \quad (19)$$

with the further restrictions $\Phi_{ij}^S = O(r^{1-\nu})$ for $4 \leq \nu < 5$ and $\Phi_{ij}^S = O(r^{-4})$ for $\nu \geq 5$;

A4:

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{1-\nu}), & \Phi &= O(r^{-\nu}), & \Phi_{ij}^A &= O(r^{-\nu}) & (\nu \geq 4, \nu \neq n), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{1-\nu}), \\ \Omega'_{ij} &= O(r^{-2}); \end{aligned} \quad (20)$$

A5:

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{1-n}), & \Phi_{ij}^A &= O(r^{-n}) & (\nu \geq n), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{1-n}), \\ \Omega'_{ij} &= O(r^{-2}). \end{aligned} \quad (21)$$

None of the above five cases can describe asymptotically flat spacetimes, cf. [3]. In cases A2–A5, the leading term falls off as $1/r^2$ at infinity and it is of type II(abd). In cases A3–A5, ℓ can be a multiple WAND. Examples in case A5 are Robinson-Trautman spacetime [22].

When $\beta_c < -2$, its precise value depends on the value of ν so that we have to consider the following possible cases.

3.1.2. Subcase (B): $\beta_c < -2$ with $\frac{n}{2} < \nu \leq 1 + \frac{n}{2}$. In this case, $\beta_c = -\frac{n}{2}$ and we have

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-n/2}), & \Phi &= O(r^{-\nu}), & \Phi_{ij}^A &= O(r^{-\nu}) & \left(\frac{n}{2} < \nu \leq 1 + \frac{n}{2}\right), \\ \Psi'_{ijk} &= O(r^{-n/2}), \\ \Omega'_{ij} &= O(r^{1-n/2}). \end{aligned} \quad (22)$$

Here, ℓ cannot be a WAND. The leading term at infinity falls off as $1/r^{n/2-1}$ and it is of type N. This includes *radiative spacetimes* [3] that are asymptotically flat in the Bondi

definition [23, 24]. If one takes for b.w. +2 components $\nu = 1 + \frac{n}{2}$ and additionally *assumes* that $\Omega_{ij} = \Omega_{ij}^{(0)} r^{-n/2-1} + \Omega_{ij}^{(1)} r^{-n/2-2} + o(r^{-n/2-2})$, then one finds [4] the peeling-off behavior

$$C_{abcd} = \frac{N_{abcd}}{r^{n/2-1}} + \frac{II_{abcd}}{r^{n/2}} + o(r^{-n/2}). \quad (23)$$

This agrees with [3] for asymptotically flat spacetimes. See [3, 4] for special properties of the case $n = 5$. When $\beta_c < -2$ but ν is not in the range $\frac{n}{2} < \nu \leq 1 + \frac{n}{2}$ one has the following subcases (B*) and (C).

3.1.3. Subcase (B):* $\beta_c < -2$ with $2 < \nu \leq \frac{n}{2}$ or $1 + \frac{n}{2} < \nu \leq n - 1$. In this case, $\beta_c = -\nu$ and we have (cf. section IV A 5 of [4])

$$\begin{aligned} \Phi_{ijkl} &= O(r^{-\nu}), & \Phi_{ij}^A &= O(r^{-\nu}), \\ \Psi'_{ijk} &= O(r^{-2}) \quad \text{if } 2 < \nu \leq 3, & \Psi'_{ijk} &= O(r^{-\nu}) \quad \text{if } \nu > 3, \\ \Omega'_{ij} &= o(r^{1-\nu}) \quad \text{if } \nu \neq \frac{n}{2}, & \Omega'_{ij} &= O(r^{1-n/2}) \quad \text{if } \nu = \frac{n}{2}. \end{aligned} \quad (24)$$

Here, ℓ cannot be a WAND.

3.1.4. Subcase (C): $\beta_c < -2$ with $\nu > n - 1$. In this case, $\beta_c = 1 - n$ and we have

$$\begin{aligned} \Phi_{ijkl} &= O(r^{1-n}), & \Phi_{ij}^A &= o(r^{1-n}) \quad (\nu > n - 1), \\ \Psi'_{ijk} &= O(r^{1-n}), \\ \Omega'_{ij} &= o(r^{2-n}), \end{aligned} \quad (25)$$

with $\Phi_{ij}^A = O(r^{-\nu})$ for $n - 1 < \nu < n$ and $\Phi_{ij}^A = O(r^{-n})$ for $\nu \geq n$. Here, ℓ can become a multiple WAND, cf. [25]. This includes asymptotically flat spacetimes in the case of *vanishing radiation* [3], such as those for which ℓ is a multiple WAND [25], e.g., the Schwarzschild-Tangherlini metric and Kerr-Schild spacetimes [26] with a non-degenerate Kerr-Schild vector.

3.2. Case (ii): $\Omega_{ij} = o(r^{-n})$, $\Psi_{ijk} = O(r^{-n})$

3.2.1. Subcase $\beta_c = -2$. Generically, one has

$$\begin{aligned} \Omega_{ij} &= o(r^{-n}), \\ \Psi_{ijk} &= O(r^{-n}), \\ \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{-4}), & \Phi_{ij}^A &= O(r^{-3}), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{-3}), \\ \Omega'_{ij} &= O(r^{-2}). \end{aligned} \quad (26)$$

For $\Psi_{ijk}^{(n)} = 0$, this case reduces to (19) (with $\nu > n$). See [4] for possible subcases.

3.2.2. Subcase $\beta_c = 1 - n$. When $\beta_c < -2$ then necessarily $\beta_c = 1 - n$ and generically, one has

$$\begin{aligned} \Omega_{ij} &= o(r^{-n}), \\ \Psi_{ijk} &= O(r^{-n}), \\ \Phi_{ijkl} &= O(r^{1-n}), & \Phi_{ij}^A &= O(r^{-n}), \\ \Psi'_{ijk} &= O(r^{1-n}), & \Psi'_i &= O(r^{1-n}), \\ \Omega'_{ij} &= o(r^{2-n}). \end{aligned} \quad (27)$$

This includes asymptotically flat spacetimes in the case of vanishing radiation [3]. For $\Psi_{ijk}^{(n)} = 0$, this case reduces to (25) (with $\nu > n$).

3.3. Case (iii): $\Omega_{ij} = o(r^{-3})$, $\Psi_{ijk} = O(r^{-3})$

This case cannot represent asymptotically flat spacetimes [3]. Generically, $\beta_c = -2$ and

$$\begin{aligned}\Omega_{ij} &= O(r^{-\nu}) & (\nu > 3), \\ \Psi_{ijk} &= O(r^{-3}), & \Psi_i = o(r^{-3}), \\ \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S = O(r^{-3}), & \Phi = o(r^{-3}), & \Phi_{ij}^A = O(r^{-3}), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i = O(r^{-3}), \\ \Omega'_{ij} &= O(r^{-2}),\end{aligned}\tag{28}$$

where $\Psi_i = O(r^{-\nu})$, $\Phi = O(r^{-\nu})$ for $3 < \nu \leq 4$ while $\Psi_i = O(r^{-4})$, $\Phi = O(r^{-4})$ for $\nu > 4$. Here, ℓ can be a single WAND and the asymptotically leading term is of type II(abd). For $\Psi_{ijk}^{(3)} = 0$, this case reduces for $3 < \nu < 4$ to (18) (with $\nu > n$), for $4 \leq \nu \leq n$ to (19) and for $\nu > n$ to (26). If $\beta_c < -2$ then $\Phi_{ijkl} = O(r^{-3})$ and the leading term at infinity becomes of type III(a).

Acknowledgments

The authors acknowledge support from research plan RVO: 67985840 and research grant GAČR 13-10042S.

References

- [1] Newman E T and Tod K P 1980 *General Relativity and Gravitation: One Hundred Years After the Birth of Albert Einstein* vol 2 ed Held A (London and New York: Plenum Press) pp 1–36
- [2] Penrose R and Rindler W 1986 *Spinors and Space-Time* vol 2 (Cambridge: Cambridge University Press)
- [3] Godazgar M and Reall H S 2012 *Phys. Rev. D* **85** 084021
- [4] Ortaggio M and Pravdová A 2014 *Phys. Rev. D* **90** 104011
- [5] Ortaggio M 2014 *Phys. Rev. D* **90** 124020
- [6] Ortaggio M, Pravda V and Pravdová A 2013 *Class. Quantum Grav.* **30** 013001
- [7] Newman E T and Penrose R 1962 *J. Math. Phys.* **3** 566–578 see also E. Newman and R. Penrose (1963), Errata, *J. Math. Phys.* 4:998.
- [8] Ortaggio M, Pravda V and Pravdová A 2007 *Class. Quantum Grav.* **24** 1657–1664
- [9] Coley A, Milson R, Pravda V and Pravdová A 2004 *Class. Quantum Grav.* **21** L35–L41
- [10] Pravda V, Pravdová A, Coley A and Milson R 2004 *Class. Quantum Grav.* **21** 2873–2897 see also V. Pravda, A. Pravdová, A. Coley and R. Milson *Class. Quantum Grav.* **24** (2007) 1691 (corrigendum).
- [11] Durkee M, Pravda V, Pravdová A and Reall H S 2010 *Class. Quantum Grav.* **27** 215010
- [12] Aliev A N and Frolov V P 2004 *Phys. Rev. D* **69** 084022
- [13] Ortaggio M and Pravda V 2006 *JHEP* **12** 054
- [14] Ortaggio M, Podolský J and Žofka M 2008 *Class. Quantum Grav.* **25** 025006
- [15] Penrose R 1963 *Phys. Rev. Lett.* **10** 66–68
- [16] Exton A, Newman E and Penrose R 1969 *J. Math. Phys.* **10** 1566–1570
- [17] van der Burg M 1969 *Proc. R. Soc. A* **310** 221–230
- [18] Aliev A N 2006 *Phys. Rev. D* **74** 024011
- [19] Emparan R 2004 *JHEP* **03** 064
- [20] Ortaggio M *Proceedings of the XVII SIGRAV Conference (Torino, September 4–7, 2006) (Preprint gr-qc/0701036)* URL <http://www.sigrav.org/Material/Torino2006/Ortaggio.pdf>
- [21] Ortaggio M, Podolský J and Žofka M 2015 *JHEP* **02** 045
- [22] Podolský J and Ortaggio M 2006 *Class. Quantum Grav.* **23** 5785–5797
- [23] Tanabe K, Tanahashi N and Shiromizu T 2010 *J. Math. Phys.* **51** 062502
- [24] Tanabe K, Kinoshita S and Shiromizu T 2011 *Phys. Rev. D* **84** 044055
- [25] Ortaggio M, Pravda V and Pravdová A 2009 *Phys. Rev. D* **80** 084041
- [26] Ortaggio M, Pravda V and Pravdová A 2009 *Class. Quantum Grav.* **26** 025008